Lecture 1: Real World Problems and Differential Equations

Goals lecture of this lecture:

1. To get a brief idea of how real world problems are converted into equations;
2. To be convinced that real world problems can be formulated into equations consisting of derivatives (Differential equations)

Goals of Math 3310

Overview on some commonly used methods for analytic solutions
(Can be used to get a rough initial guess of the solution)

Modern numerical methods for the approximation of solutions

Main idea

Main tasks of Applied Mathematics

Real world problems

↓

Rules + physical phenomenons + problem requirements
(Mathematicians communicate with “customers”)

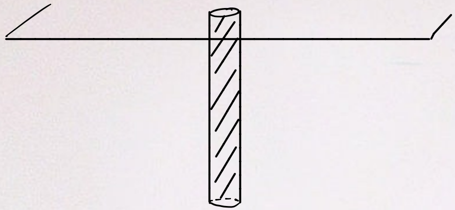
↓

Mathematics formulation
(e.g. infinitesimal analysis, energy minimisation)

↓

Solving differential equations (Main goal of Math 3310)

Example 1: Elastic bar

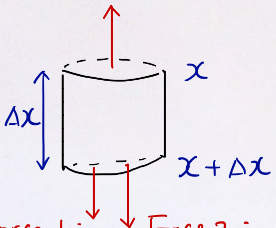


Goal: Model the displacement $u(x)$ of the elastic bar at each position x under gravity.

(Elastic bar hanged vertically under gravity)

Talk to "customers" (physicists):

Force : $c(x)\,\frac{du}{dx}(x)$



(A small portion of the elastic bar)

Force 1 : $\rho\,\Delta x\,a$

Force 2 : $c(x+\Delta x)\,\frac{du}{dx}(x+\Delta x)$

Force 1 = gravitational force

$$= \underbrace{\rho}_{\text{density}}\,(\Delta x\,\underbrace{a}_{\text{cross-sectional area}})$$

At equilibrium state, all forces will be balanced.

$$\therefore \quad c(x+\Delta x)\frac{du}{dx}(x+\Delta x) - c(x)\frac{du}{dx}(x) + \left(\rho(x)\Delta x\, a\right) g = 0$$

Turn this formulation into an equation by dividing both sides by Δx and take $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} \frac{c(x+\Delta x)\frac{du}{dx}(x+\Delta x) - c(x)\frac{du}{dx}(x)}{\Delta x} + \rho(x)\, a g = 0$$

or $$\left[\lim_{\Delta x \to 0} \frac{c(x)\frac{du}{dx}\Big|_{x+\Delta x} - c(x)\frac{du}{dx}\Big|_{x}}{\Delta x}\right] + \rho(x)\, a g = 0$$

$$\text{"} \quad \frac{d}{dx}\left(c(x)\frac{du}{dx}\right)$$

$$\Leftrightarrow \quad -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = \underbrace{\rho(x)\, a g}_{f(x)} \qquad \text{(Differential equation!)}$$

Is a differential equation enough to determine the solution?

Consider a simple case. Let $c(x) \equiv 1$ and $f(x) = x^2 + 1$.

Then:

$$-\frac{d}{dx}\left(c(x)\,\frac{d\,u(x)}{dx}\right) = x^2 + 1$$

$$\int -\frac{d}{dx}\left(\frac{d\,u(x)}{dx}\right)dx = \int (x^2 + 1)\,dx$$

$$\int -\frac{du}{dx} = \int \frac{x^3}{3} + x + C$$

$$-u(x) = \frac{x^4}{12} + \frac{x^2}{2} + Cx + D$$

Solution cannot be determined as it involves two unknown variables. Need more conditions!

What if we know $u(0) = u(1) = 0$ (fixing the two end points)

Then: $$u(x) = -\frac{x^4}{12} - \frac{x^2}{2} - Cx - D$$

$$\Rightarrow \quad u(0) = -D = 0 \quad \Rightarrow \quad D = 0$$

$$u(1) = -\frac{1}{12} - \frac{1}{2} - C = 0 \quad \Rightarrow \quad C = \frac{-7}{12}$$

$$\therefore \quad u(x) = -\frac{x^4}{12} - \frac{x^2}{2} + \frac{7}{12} \quad \text{(unique sol)}$$

What if we know: $u(0) = 0$ and $\left.\dfrac{du}{dx}\right|_{x=1} = 0$.

Then:
$$u(x) = -\frac{x^4}{12} - \frac{x^2}{2} - Cx - D$$
$$\frac{du}{dx}(x) = -\frac{x^3}{3} - x - C$$

$$\Rightarrow \quad u(0) = -D = 0 \quad \Rightarrow \quad D = 0$$
$$\frac{du}{dx}(1) = -\frac{1}{3} - 1 - C = 0 \quad \Rightarrow \quad C = -\frac{4}{3}.$$

$$\therefore \quad u(x) = -\frac{x^4}{12} - \frac{x^2}{2} + \frac{4}{3}x \quad (\text{Unique solution})$$

To determine a unique solution, we need more conditions!
(Need to ask "customers" what happens on the boundaries.)

A. Dirichlet : $u(0) = c_1$ and $u(1) = c_2$

(Does NOT involve derivatives)

B. Dirichlet + Neumann :

$$u(0) = c_1 \quad \text{and} \quad c(x) \frac{du}{dx}\bigg|_{x=1} = c_2$$

(Neumann = involves derivatives)

Example 2: (Smooth approximation of unsmooth measurement)

Goal: Given a function (measurement) $w : [0,1] \to \mathbb{R}$, which is unsmooth. Find a smooth approximation $u : [0,1] \to \mathbb{R}$ of w, such that $u(0) = w(0) = 0$.

Rule: Unsmooth means $\left|\frac{du}{dx}\right|$ is big!

Mathematical formulation: (*)

Find $u : [0,1] \to \mathbb{R}$ with $u(0) = 0$ such that:

$$J(u) = \int_0^1 \left|\frac{du}{dx}\right| dx + \underbrace{\int_0^1 (u(x) - w(x))^2}_{\substack{u(x) \text{ is close to } w(x)\\ \text{(good approximation)}}} \quad \text{is minimized}$$

$$\left|\frac{du}{dx}\right| \approx \underbrace{\sqrt{\left(\frac{du}{dx}\right)^2 + \underset{\substack{\uparrow\\ \text{small}}}{\varepsilon^2}}}_{\text{Smoothness}}$$

How to solve $(*)$?

?? We haven't learnt minimization over function $u(x)$!! ??

A smart trick:

Suppose u is the minimizer of J.

Add $u(x)$ by any small pertubation $v : [0,1] \to \mathbb{R}$ with $v(0) = 0$ to get a new function $u + tv : [0,1] \to \mathbb{R}$ (for $t \in \mathbb{R}$).

Note that $(u+tv)(0) = \underset{0}{\underset{\shortparallel}{u(0)}} + t\,\underset{0}{\underset{\shortparallel}{v(0)}} = 0$

($\therefore$ satisfies the condition)

Let $G(t) \overset{\text{def}}{=} J(u+tv) = \int_0^1 \sqrt{\left(\frac{d}{dx}(u+tv)\right)^2 + \varepsilon^2}\, dx + \int_0^1 \left[(u+tv) - w\right]^2 dx$

Then: $G : \mathbb{R} \to \mathbb{R}$ depends on $t \in \mathbb{R}$.

In particular, $G(0) = J(u) =$ minimum of J.

$\therefore$ G attains minimum at $t = 0$. $\therefore$ $\left.\dfrac{dG}{dt}\right|_{t=0} = G'(0) = 0$.

$$\therefore \left.\frac{d}{dt}\right|_{t=0} G(t) = 0 = \left.\frac{d}{dt}\right|_{t=0}\left(\int_0^1 \sqrt{\left(\frac{d}{dx}(u+tv)\right)^2 + \varepsilon^2}\, dx + \int_0^1 \left[(u+tv) - w\right]^2 dx\right)$$

$$= \int_0^1 \left.\frac{\left(\frac{du}{dx} + t\frac{dv}{dx}\right)\frac{dv}{dx}}{\sqrt{\left(\frac{d}{dx}(u+tv)\right)^2 + \varepsilon^2}}\right|_{t=0} dx + 2\int_0^1 \left.(u+tv-w)v\right|_{t=0} dx$$

$$= \int_0^1 \frac{\left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right)}{\sqrt{\left(\frac{du}{dx}\right)^2 + \varepsilon^2}} dx + 2\int_0^1 (u-w)v\, dx$$

$$\therefore\ 0 = \left.\frac{dG}{dt}\right|_{t=0} = \int_0^1 \frac{\left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right)}{\sqrt{\left(\frac{du}{dx}\right)^2 + \varepsilon^2}}\,dx + 2\int_0^1 (u-w)\,v\,dx$$

(integration by part)

$$= \int_0^1 -\frac{d}{dx}\left(\frac{\left(\frac{du}{dx}\right)}{\sqrt{\left(\frac{du}{dx}\right)^2+\varepsilon^2}}\right)v(x)\,dx + \left.\frac{\left(\frac{du}{dx}\right)v(x)}{\sqrt{\left(\frac{du}{dx}\right)^2+\varepsilon^2}}\right|_{x=0}^{x=1}$$

$$+ 2\int_0^1 \big(u(x) - w(x)\big)\,v(x)\,dx$$

Since $v(x)$ is arbitrary, we take $v(x)$ such that $v(0) = v(1) = 0$.

Then: $$\int_0^1 \left[-\frac{d}{dx}\left(\frac{\left(\frac{du}{dx}\right)}{\sqrt{\left(\frac{du}{dx}\right)^2+\varepsilon^2}}\right) + 2\big(u(x) - w(x)\big)\right] v(x)\,dx = 0$$

for all $v(x)$.

Rule: If $\int_0^1 f(x)\, g(x)\, dx = 0$ for all $g(x)$ with $g(0) = g(1) = 0$,

then $f(x) \equiv 0$.

(Let $g(x) = f(x)$. Then: $\int_0^1 f(x) \overset{\overset{f(x)}{\shortparallel}}{g(x)} = 0$

$\Rightarrow \int_0^1 \underbrace{(f(x))^2}_{\text{positive}} dx = 0 \Rightarrow f(x) \equiv 0$)

In our case,

$$\int_0^1 \left[-\frac{d}{dx}\left(\frac{\left(\frac{du}{dx}\right)}{\sqrt{\left(\frac{du}{dx}\right)^2 + \varepsilon^2}} \right) + 2(u(x) - w(x)) \right] v(x)\, dx = 0$$

for all $v(x)$.

$$\Rightarrow \quad -\frac{d}{dx}\left(\frac{\left(\frac{du}{dx}\right)}{\sqrt{\left(\frac{du}{dx}\right)^2 + \varepsilon^2}} \right) + 2(u(x) - w(x)) = 0$$

(Differential equation)

Boundary conditions:

$$u(0) = 0 \quad ; \quad \left.\frac{du}{dx}\right|_{x=1} = 0 .$$